

Local Behavior of Airy Processes

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Abstract

The Airy processes describe spatial fluctuations in wide range of growth models, where each particular Airy process arising in each case depends on the geometry of the initial profile. We show how the coupling method, developed in the last-passage percolation context, can be used to prove that several types of Airy processes have a continuous version, and behave locally like a Brownian motion.

1 Introduction

In [6] the coupling method was applied to study local fluctuations of point to point last-passage percolation (LPP) times and its scaling limit, the Airy_2 process. This method relies on a local comparison lemma that allows us to sandwich local fluctuations of point to point LPP times in between Brownian local fluctuations of the equilibrium regime. The local comparison lemma can be seen as a property involving a basic coupling of LPP models started from wedge (point-to-point) and equilibrium initial profiles. The aim is to use an extension of this property for two arbitrary initial profiles, to show Brownian local fluctuations for several types of Airy processes, where each particular Airy process arising in each case depends on the geometry of initial profile.

1.1 Local fluctuations of the LPP model

We start with some definitions. For $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^2$ we define that $\mathbf{x} \leq \mathbf{y}$, for $\mathbf{x} = (i, j) \in \mathbb{Z}^2$ and $\mathbf{y} = (k, l)$, if $i \leq k$ and $j \leq l$. We say that a sequence $\pi = (\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n)$ is an up-right path from \mathbf{x} to \mathbf{y} , with $\mathbf{x} \leq \mathbf{y}$, if $\mathbf{x}_{m+1} - \mathbf{x}_m \in \{(1, 0), (0, 1)\}$, $\mathbf{x}_0 = \mathbf{x}$ and $\mathbf{x}_n = \mathbf{y}$. We denote $\Pi(\mathbf{x}, \mathbf{y})$ the set of all up-right paths from \mathbf{x} to \mathbf{y} . The random environment in our setting is given by a collection $\omega \equiv \{\omega_{i,j} : i + j > 0\}$ of i.i.d. random variables (passage times) with exponential (or geometric) distribution of parameter 1. Given $\pi = (\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n) \in \Pi(\mathbf{x}, \mathbf{y})$ the passage time is set as

$$\omega(\pi) := \sum_{i=1}^n \omega_i.$$

Notice that we do not include the passage time at $\mathbf{x}_0 = \mathbf{x}$. For $\mathbf{x} \leq \mathbf{y}$, with $\mathbf{x} = (i, j)$ and $i + j > 0$, the point-to-point last-passage percolation time is defined as

$$L(\mathbf{x}, \mathbf{y}) := \max_{\pi \in \Pi(\mathbf{x}, \mathbf{y})} \omega(\pi).$$

Denote $L(\mathbf{x}) \equiv L(\mathbf{0}, \mathbf{x})$, $L_k(\mathbf{x}) \equiv L((k, -k), \mathbf{x})$ and set

$$C^{\mathbf{x}} := \{k \in \mathbb{Z} : (k, -k) \leq \mathbf{x}\}.$$

Given a profile

$$b : \mathbb{Z} \rightarrow \mathbb{Z} \cup \{-\infty\} \text{ with } b(0) = 0,$$

the last-passage percolation time with initial boundary profile b is defined as

$$L^b(\mathbf{x}) := \max_{k \in C^{\mathbf{x}}} \{b(k) + L_k(\mathbf{x})\}.$$

The last-passage percolation model generates a discrete time Markov process $(M_n^b)_{n \geq 0}$ defined as

$$M_n^b(k) := L^b[k]_n - L^b[0]_n, \text{ for } k \in \mathbb{Z}.$$

(Note that $M_0^b \equiv b$.) The increments $L^b[k]_n - L^b[0]_n$ are defined along $[k]_n = (k+n, k-n)$, so that n represents the time parameter. For a real number $x \in [-n, n]$ we denote

$$[x]_n \equiv [\lfloor x \rfloor]_n \equiv (n + \lfloor x \rfloor, n - \lfloor x \rfloor).$$

For a fixed constant $C > 0$, define the process

$$\Delta_n^b(u) := \frac{L^b[un^{2/3}]_n - L^b[0]_n}{2^{3/2}n^{1/3}}, u \in [0, C]$$

(where $n \geq C^3$). We could have defined a continuous one by linearly interpolating the values, but this would not be relevant for our purposes. Our goal is to prove tightness of Δ_n^b and a local functional convergence of any weak limit to Brownian motion.

Let

$$X(\rho) = \text{Exp}_1(1 - \rho) - \text{Exp}_2(\rho),$$

where $\text{Exp}_1(1 - \rho)$ and $\text{Exp}_2(\rho)$ are independent random variables with exponential distributions of parameter $1 - \rho$ and ρ , respectively. The unique family of time stationary measure s (with ergodic space increments) for the LPP Markov process is the one induced by a sum of i.i.d. copies ζ_k of $X(\rho)$. If one sets $s(0) = 0$ and

$$s_\rho(k) = \begin{cases} \sum_{i=k+1}^0 -\zeta_i & \text{for } k < 0 \\ 0 & \text{for } k = 0 \\ \sum_{i=1}^k \zeta_i & \text{for } k > 0, \end{cases}$$

then [1]

$$M_n^{s_\rho} \stackrel{\text{dist.}}{=} s_\rho, \forall n \geq 0. \quad (1.1)$$

As a corollary of the functional central limit theorem for sums of i.i.d. random variables, in the stationary regime with $\rho = 1/2$, we have that

$$\lim_{n \rightarrow \infty} \Delta_n^{s_{1/2}} \stackrel{\text{dist.}}{=} B, \quad (1.2)$$

in the Skorohod topology of cadlag functions on compact sets, where B is a standard Brownian Motion. As we shall see next, we will be able to study Δ_n^b by comparing its local behavior with the stationary regime.

To state the main result of this paper we need to introduce the exit-point location,

$$Z^b(\mathbf{x}) := \arg \max_{k \in C_{\mathbf{x}}} \{b(k) + L_k(\mathbf{x})\}$$

(which is a.s. unique), in such a way that,

$$L^b(\mathbf{x}) = b\left(Z^b(\mathbf{x})\right) + L_{Z^b(\mathbf{x})}(\mathbf{x}).$$

The exit point location is the key to compare the evolution of M_n^b with the equilibrium regime $M_n^{s_{1/2}}$ (see Lemma 2.1). To keep the evolution close enough to equilibrium, we will need the following assumption.

Assumption 1 *Let $C \geq 0$ and for $r \geq 0$ define*

$$R_C(r) := \limsup_{n \rightarrow \infty} \mathbb{P}\left(|Z^b[Cn^{2/3}]_n| \geq rn^{2/3}\right).$$

Then

$$\limsup_{r \rightarrow \infty} R_C(r) = 0.$$

Theorem 1 *Under Assumption 1, the collection $\{\Delta_n^b : n \geq 1\}$ is tight in the Skorohod topology of cadlag functions on compact sets, and any weak limit¹ is continuous almost surely. Furthermore, for any weak limit Δ^b of $\{\Delta_n^b : n \geq 1\}$ we have that*

$$\lim_{\epsilon \downarrow 0} \epsilon^{-1/2} \Delta^b(\epsilon x) \stackrel{dist.}{=} B(x), \quad (1.3)$$

in the topology of continuous functions on compact sets, where $(B(x), x \in \mathbb{R})$ is a standard two-sided Brownian Motion.

Remark 1 *In a forthcoming paper we will apply the coupling method to prove ergodicity of the KPZ fixed point [10] and decorrelation at large times. These ideas will also be use to show that the Airy Sheet is locally an additive Brownian Motion.*

1.2 Examples of initial profiles

1.2.1 Narrow Wedge Profile

If

$$w(k) = \begin{cases} 0 & \text{for } k = 0 \\ -\infty & \text{for } k \neq 0, \end{cases}$$

then

$$L^w(\mathbf{x}) = L(\mathbf{x}) \text{ and } Z^w(\mathbf{x}) \equiv 0.$$

Define

$$H_n^w(u) = \frac{L^w[2^{2/3}un^{2/3}]_n - 4n}{2^{4/3}n^{1/3}}, u \in [0, C].$$

¹Which existence is ensured by Prohorov's theorem.

Then

$$H_n^w(u) = H_n^w(0) + 2^{1/6} \Delta_n^w(2^{2/3}u).$$

The Airy_2 process arise as the limit of A_n^w [11, 2]:

$$\lim_{n \rightarrow \infty} H_n^w(u) \stackrel{\text{dist.}}{=} A_2(u) - u^2,$$

Clearly Assumption 1 is true in this case, and we can use Theorem 1 to show tightness of H_n^w and local (functional) convergence of the Airy_2 process to Brownian Motion:

$$\lim_{\epsilon \downarrow 0} \epsilon^{-1/2} (A_2(\epsilon x) - A_2(0)) \stackrel{\text{dist.}}{=} \sqrt{2}B(x).$$

This was actually the main result in [6], in a slightly different context, where the local comparison was use for the first time. Different approaches to prove local convergence were developed in [8] and in [7].

1.2.2 Flat Profile

The flat profile is defined as $f(k) = 0$ for all $k \in \mathbb{Z}$ (line to point last-passage percolation). Define

$$H_n^f(u) = \frac{L^f[2^{2/3}un^{2/3}]_n - \mu n}{2^{4/3}n^{1/3}}, u \in [0, C].$$

Then

$$H_n^f(u) = H_n^f(0) + 2^{1/6} \Delta_n^f(2^{2/3}u).$$

The Airy_1 process arise as the limit of H_n^f :

$$\lim_{n \rightarrow \infty} H_n^f(u) \stackrel{\text{dist.}}{=} A_1(u),$$

in the sense of convergence of finite-dimensional distributions [14, 3]. Assumption 1 is true in this case as well.

Lemma 1.1 *Assumption 1 holds for the flat profile.*

Theorem 1, together with Lemma 1.1, implies tightness of H_n^f , and local (functional) convergence of the Airy_1 process to Brownian Motion:

$$\lim_{\epsilon \downarrow 0} \epsilon^{-1/2} (A_1(\epsilon x) - A_1(0)) \stackrel{\text{dist.}}{=} \sqrt{2}B(x).$$

We believe that this the first work that brings tightness of $A_{1,n}$ and functional convergence of local fluctuations of A_1 . Finite dimensional convergence of the Airy_1 was first prove in [12]. We note that

$$\mathbb{P} \left(\max_{k \in [-rn^{2/3}, rn^{2/3}]} \{L_k[0]_n\} \neq L^f[0]_n \right) = \mathbb{P} \left(|Z^f[0]_n| > rn^{2/3} \right).$$

Therefore, Lemma 1.1 implies that

$$\lim_{n \rightarrow \infty} \frac{Z^f[0]_n}{2^{2/3}n^{2/3}} \stackrel{\text{dist.}}{=} \arg \max_u \{A_2(u) - u^2\}. \quad (1.4)$$

See Proposition 1.4 and Theorem 1.6 in [9] for and analog results for LPP with geometric weights.

1.2.3 Mixed Profiles

Mixed initial profiles can be obtained by placing one condition on each half of \mathbb{Z} :

$$\begin{aligned} \text{wf}(k) &= \begin{cases} -\infty & \text{for } k < 0 \\ 0 & \text{for } k \geq 0, \end{cases} \quad (\text{wedge to flat}) \\ \text{ws}(k) &= \begin{cases} -\infty & \text{for } k < 0 \\ 0 & \text{for } k = 0 \\ \sum_{i=1}^k \zeta_i & \text{for } k > 0, \end{cases} \quad (\text{wedge to stationary}) \\ \text{fs}(k) &= \begin{cases} 0 & \text{for } k \leq 0 \\ \sum_{i=1}^k \zeta_i & \text{for } k > 0. \end{cases} \quad (\text{flat to stationary}) \end{aligned}$$

For each case there is a specific Airy process, denoted $\text{Airy}_{2 \rightarrow 1}$ (wedge to flat), $\text{Airy}_{2 \rightarrow 0}$ (wedge to stationary) and $\text{Airy}_{1 \rightarrow 0}$ (flat to stationary).

Lemma 1.2 *Assumption 1 holds for all mixed profiles.*

Theorem 1, together with Lemma 1.2, implies tightness of each rescaled processes, and local (functional) convergence to Brownian Motion of the respective Airy process. We believe that this is the first work that brings functional convergence of local fluctuations for mixed profiles. We note again that Lemma 1.2 implies that the maximum is attained in a compact set with high probability, which allow us to related the distribution of the exit point with a variational problem involving the Airy_2 process (as in (1.4)).

Remark 2 *For any initial profiles that produces a rarefaction fan (wedge type of profile), the exit point will stabilize close to origin. This can be used to prove Assumption 1 and, as a consequence, to apply Theorem 1 in this context as well.*

2 Exit Points and Local Comparison

Given two profiles b_1 and b_2 , the basic coupling (L^{b_1}, L^{b_2}) is constructed by setting

$$L^{b_i}(\mathbf{x}) := \max_{k \in C_{\mathbf{x}}} \{b_i(k) + L_k(\mathbf{x})\}.$$

(Recall that L is a function of the same environment ω). In [6], the key result to study the local fluctuations of L is given as follows (see Lemma 1 in [6] for the LPP-Poissonian version). Let $k \leq l$ and $n \geq 1$. If $Z^b[k]_n \geq 0$ then

$$L[l]_n - L[k]_n \leq L^b[l]_n - L^b[k]_n,$$

and if $Z^b[l]_n \leq 0$ then

$$L[l]_n - L[k]_n \geq L^b[l]_n - L^b[k]_n.$$

As we noted before, for the wedge profile we have that $L^w = L$ and $Z^w(\mathbf{x}) \equiv 0$, and the above inequalities can be seen as a comparison of local increments of the LPP model with respect to w and b , in terms of the relative location of the respective exit-points. This local comparison property can be generalize for the coupling (L^{b_1}, L^{b_2}) , with arbitrary b_1 and b_2 , as follows.

Lemma 2.1 *Let $k \leq l$ and $n \geq 1$. If $Z^{b_1}[l]_n \leq Z^{b_2}[k]_n$ then*

$$L^{b_1}[l]_n - L^{b_1}[k]_n \leq L^{b_2}[l]_n - L^{b_2}[k]_n.$$

Proof Let $\pi(\mathbf{x}, \mathbf{y})$, for $\mathbf{x} \leq \mathbf{y}$, denote the path which attains the last-passage percolation time:

$$\pi(\mathbf{x}, \mathbf{y}) = \arg \max_{\pi \in \Pi(\mathbf{x}, \mathbf{y})} \omega(\pi),$$

so that $L(\mathbf{x}, \mathbf{y}) = \omega(\pi(\mathbf{x}, \mathbf{y}))$. Notice that

$$L(\mathbf{x}, \mathbf{y}) = L(\mathbf{x}, \mathbf{z}) + L(\mathbf{z}, \mathbf{y}),$$

for any $\mathbf{z} \in \pi(\mathbf{x}, \mathbf{y})$. Denote $z_1 \equiv Z^{b_1}[l]_n$ and $z_2 \equiv Z^{b_2}[k]_n$ so that, by assumption, $z_1 \leq z_2$. Let \mathbf{c} be a crossing between $\pi([z_1]_0, [l]_n)$ and $\pi([z_2]_0, [k]_n)$. Such a crossing always exists because $k \leq l$ and $z_1 \leq z_2$. We remark that, by superadditivity,

$$L^{b_2}[l]_n \geq b_2(z_2) + L([z_2]_0, [l]_n) \geq b_2(z_2) + L([z_2]_0, \mathbf{c}) + L(\mathbf{c}, [l]_n).$$

We use this, and that (since $\mathbf{c} \in \pi([z_2]_0, [k]_n)$)

$$b_2(z_2) + L([z_2]_0, \mathbf{c}) - L^{b_2}[k]_n = -L(\mathbf{c}, [k]_n),$$

in the following inequality:

$$\begin{aligned} L^{b_2}[l]_n - L^{b_2}[k]_n &\geq b_2(z_2) + L([z_2]_0, \mathbf{c}) + L(\mathbf{c}, [l]_n) - L^{b_2}[k]_n \\ &= L(\mathbf{c}, [l]_n) - L(\mathbf{c}, [k]_n). \end{aligned}$$

By superadditivity,

$$-L(\mathbf{c}, [k]_n) \geq L^{b_1}(\mathbf{c}) - L^{b_1}[k]_n,$$

and hence (since $\mathbf{c} \in \pi([z_1]_0, [l]_n)$)

$$\begin{aligned} L^{b_2}[l]_n - L^{b_2}[k]_n &\geq L(\mathbf{c}, [l]_n) - L(\mathbf{c}, [k]_n) \\ &\geq L(\mathbf{c}, [l]_n) + L^{b_1}(\mathbf{c}) - L^{b_1}[k]_n \\ &= L^{b_1}[l]_n - L^{b_1}[k]_n. \end{aligned}$$

□

Now the aim is to compare the local increment corresponding to a given profile b with the local increment of the stationary profile s_ρ . From now on we will put a superscript ρ for quantities related to the stationary measure, such as $L^\rho \equiv L^{s_\rho}$ and $Z^\rho \equiv Z^{s_\rho}$. Due to the (space) stationarity of the increments of s_ρ , the location of the exit-point satisfies:

$$Z^\rho[k+h]_n \stackrel{dist.}{=} Z^\rho[k]_n + h.$$

To control the fluctuations of Z^ρ one has to look at the so called characteristic of the system given by the direction $(a_\rho, 1)$, where

$$a_\rho = \left(\frac{1-\rho}{\rho} \right)^2.$$

In the anti-diagonal $x + y = 2n$, this corresponds to $x = n + k$ where

$$k = b_\rho n = \left(\frac{a_\rho - 1}{a_\rho + 1} \right) n.$$

Along this direction the exit-point oscillates around the origin in the $n^{2/3}$.

Lemma 2.2 *There exists a constant $c_1 > 0$ such that, for all $\rho \in [1/4, 3/4]$*

$$\limsup_{n \rightarrow \infty} \mathbb{P} \left(|Z^\rho[b_\rho n]_n| \geq r n^{2/3} \right) \leq \frac{c_1}{r^3},$$

for all $r > 1$.

Proof If we denote \tilde{Z}^ρ to be the exit-point with respect to the positive coordinate axis then

$$\left\{ |Z^\rho[b_\rho n]_n| \geq r n^{2/3} \right\} \subseteq \left\{ |\tilde{Z}^\rho[b_\rho n]_n| \geq r n^{2/3} \right\}.$$

Thus, Lemma 2.2 follows from Theorem 2.2 in [1]. □

The reason we need an uniform control for all $\rho \in [1/4, 3/4]$ is that we will scale ρ with n in a neighborhood of $1/2$. For this neighborhood we have $0 \leq a_\rho \leq 9$ and so

$$0 \leq \frac{1}{10}(a_\rho - 1) \leq b_\rho, \text{ for } \rho \in [1/4, 1/2],$$

and

$$0 \geq \frac{1}{10}(a_\rho - 1) \geq b_\rho, \text{ for } \rho \in [1/2, 3/4].$$

We note that a_ρ is a decreasing function of ρ and its derivative is bounded in the interval $[1/4, 3/4]$. So by using the mean value theorem, one can see that there are constants $c_2, c_3 \in (0, \infty)$ such that

$$0 \leq -c_2 \left(\rho - \frac{1}{2} \right) \leq b_\rho, \text{ for } \rho \in [1/4, 1/2],$$

and

$$0 \geq -c_3 \left(\rho - \frac{1}{2} \right) \geq b_\rho, \text{ for } \rho \in [1/2, 3/4].$$

We will be interested in the regime

$$\rho_n^\pm = 1/2 \pm \frac{r}{n^{1/3}},$$

where r is fixed and n is large enough so that $\rho_n^\pm \in [1/4, 3/4]$. Therefore,

$$0 \leq c_2 r n^{2/3} \leq b_{\rho_n^-} n,$$

and

$$0 \geq -c_3 r n^{2/3} \geq b_{\rho_n^+} n.$$

Lemma 2.3 For $r, C > 0$ set

$$\rho_n^\pm = \rho_n^\pm(r) = 1/2 \pm \frac{r}{n^{1/3}},$$

and define the event

$$E_n(r) := \left\{ Z^{\rho_n^-}[Cn^{2/3}]_n \leq Z^b[0]_n \text{ and } Z^b[Cn^{2/3}]_n \leq Z^{\rho_n^+}[0]_n \right\}.$$

Under Assumption 1,

$$\limsup_{r \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}(E_n(r)^c) = 0.$$

Proof Let us first show that

$$\limsup_{r \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}\left(Z^b[Cn^{2/3}]_n > Z^{\rho_n^+}[0]_n\right) = 0.$$

Since,

$$\mathbb{P}\left(Z^b[Cn^{2/3}]_n > Z^{\rho_n^+}[0]_n\right) \leq \mathbb{P}\left(|Z^b[Cn^{2/3}]_n| > \frac{c_3}{2}rn^{2/3}\right) + \mathbb{P}\left(Z^{\rho_n^+}[0]_n < \frac{c_3}{2}rn^{2/3}\right),$$

By Assumption 1, we only need to control

$$\limsup_{n \rightarrow \infty} \mathbb{P}\left(Z^{\rho_n^+}[0]_n < \frac{c_3}{2}rn^{2/3}\right).$$

On the other hand, $Z^{\rho_n^+}[0]_n \stackrel{dist.}{=} Z^{\rho_n^+}[b_{\rho_n^+}n]_n - b_{\rho_n^+}n$, and so

$$\begin{aligned} \mathbb{P}\left(Z^{\rho_n^+}[0]_n < \frac{c_3}{2}rn^{2/3}\right) &= \mathbb{P}\left(Z^{\rho_n^+}[b_{\rho_n^+}n]_n < \frac{c_3}{2}rn^{2/3} + b_{\rho_n^+}n\right) \\ &\leq \mathbb{P}\left(Z^{\rho_n^+}[b_{\rho_n^+}n]_n < \left(\frac{c_3}{2} - c_3\right)rn^{2/3}\right) \\ &= \mathbb{P}\left(Z^{\rho_n^+}[b_{\rho_n^+}n]_n < -\frac{c_3}{2}rn^{2/3}\right) \\ &\leq \mathbb{P}\left(|Z^{\rho_n^+}[b_{\rho_n^+}n]_n| > \frac{c_3}{2}rn^{2/3}\right). \end{aligned}$$

By Lemma 2.2, this shows that

$$\limsup_{n \rightarrow \infty} \mathbb{P}\left(Z^{\rho_n^+}[0]_n < \frac{c_3}{2}n^{2/3}\right) \leq \frac{c'_1}{r^3},$$

for large enough r . To estimate

$$\limsup_{n \rightarrow \infty} \mathbb{P}\left(Z^b[Cn^{2/3}]_n < Z^{\rho_n^-}[0]_n\right),$$

we use a similar argument:

$$\mathbb{P}\left(Z^b[0]_n < Z^{\rho_n^-}[Cn^{2/3}]_n\right) \leq \mathbb{P}\left(|Z^b[0]_n| > \frac{c_2}{2}rn^{2/3}\right) + \mathbb{P}\left(Z^{\rho_n^-}[Cn^{2/3}]_n > -\frac{c_2}{2}rn^{2/3}\right),$$

and

$$\begin{aligned}\mathbb{P}\left(Z^{\rho_n^-}[Cn^{2/3}]_n > -\frac{c_2}{2}rn^{2/3}\right) &= \mathbb{P}\left(Z^{\rho_n^-}[b_{\rho_n^-}n]_n > -\frac{c_2}{2}rn^{2/3} + b_{\rho_n^-}n - Cn^{2/3}\right) \\ &\leq \mathbb{P}\left(Z^{\rho_n^+}[b_{\rho_n^+}n]_n > \left(\frac{c_2}{2}r - C\right)n^{2/3}\right).\end{aligned}$$

Hence (by Lemma 2.2)

$$\limsup_{n \rightarrow \infty} \mathbb{P}\left(Z^{\rho_n^-}[Cn^{2/3}]_n > -\frac{c_2}{2}rn^{2/3}\right) \leq \frac{c_1''}{r^3},$$

for large enough r . □

The stationary profile with parameter ρ has mean

$$m_\rho = \mathbb{E}(X(\rho)) = \mathbb{E}(\text{Exp}_1(1 - \rho)) - \mathbb{E}(\text{Exp}_2(\rho)) = \frac{2\rho - 1}{\rho(1 - \rho)}.$$

Thus, $4(2\rho - 1) \leq m_\rho \leq 6(2\rho - 1)$, for all $\rho \in [1/4, 3/4]$, and

$$m_{\rho_n^+}n^{1/3} \leq 6r \quad \text{and} \quad m_{\rho_n^-}n^{1/3} \geq -6r, \quad (2.1)$$

for large enough n (and fixed $r > 0$). Let

$$B_n^\pm(u) := \frac{L^{\rho_n^\pm}[un^{2/3}] - L^{\rho_n^\pm}[0] - m_{\rho_n^\pm}un^{2/3}}{2^{3/2}n^{1/3}}.$$

Lemma 2.4 *On the event $E_n(r)$,*

$$B_n^-(v) - B_n^-(u) - 3\sqrt{2}(v - u)r \leq \Delta_n^b(v) - \Delta_n^b(u) \leq B_n^+(v) - B_n^+(u) + 3\sqrt{2}(v - u)r,$$

for all $u, v \in [0, C]$.

Proof For fixed $n \geq 1$, $Z^b[k]_n$ is a nondecreasing function of k and, on the event $E_n(r)$,

$$Z^{\rho_n^-}[vn^{2/3}]_n \leq Z^{\rho_n^-}[Cn^{2/3}]_n \leq Z^b[0]_n \leq Z^b[un^{2/3}]_n,$$

and

$$Z^b[vn^{2/3}]_n \leq Z^b[Cn^{2/3}]_n \leq Z^{\rho_n^+}[0]_n \leq Z^{\rho_n^+}[un^{2/3}]_n,$$

for all $u, v \in [0, C]$. By Lemma 2.1, on the event $E_n(r)$,

$$L^{\rho_n^-}[vn^{2/3}]_n - L^{\rho_n^-}[un^{2/3}]_n \leq L^b[vn^{2/3}]_n - L^b[un^{2/3}]_n \leq L^{\rho_n^+}[vn^{2/3}]_n - L^{\rho_n^+}[un^{2/3}]_n,$$

which yields to

$$B_n^-(v) - B_n^-(u) + 2^{-3/2}m_{\rho_n^-}n^{1/3} \leq \Delta_n^b(v) - \Delta_n^b(u) \leq B_n^+(v) - B_n^+(u) + 2^{-3/2}m_{\rho_n^+}n^{1/3},$$

for all $u, v \in [0, C]$. Together with (2.1), this proves Lemma 2.4 □

The modulus of continuity of a process $(X(u), u \in [0, C])$ is defined as

$$W_X(\delta) := \sup_{u, v \in [0, C], |u - v| \leq \delta} |X(u) - X(v)|, \quad \text{where } \delta \in [0, C].$$

Corollary 2.1 *On the event $E_n(r)$,*

$$W_{\Delta_n^b}(\delta) \leq \max \left\{ W_{B_n^-}(\delta), W_{B_n^+}(\delta) \right\} + 3\sqrt{2}\delta r.$$

Proof of Theorem 1. Denote

$$R_1(r) := \limsup_{n \rightarrow \infty} \mathbb{P}(E_n(r)^c).$$

We will use Theorem 15.5 in [4]. It states that if

- $\limsup_{a \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}(|X_n(0)| > a) = 0$;
- for every $\eta > 0$, $\limsup_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}(W_{X_n}(\delta) > \eta) = 0$;

then $\{X_n : n \geq 1\}$ is tight in the Skorohod topology of cadlag functions on compact sets, and any weak limit is continuous almost surely. Since $\Delta_n^b(0) = 0$ we only need to check the second condition: by Corollary 2.1,

$$\mathbb{P}(W_{\Delta_n^b}(\delta) > \eta) \leq \mathbb{P}(W_{B_n^-}(\delta) > \eta - 3\sqrt{2}\delta r) + \mathbb{P}(W_{B_n^+}(\delta) > \eta - 3\sqrt{2}\delta r) + \mathbb{P}(E_n(r)^c).$$

Clearly $n^{2/3} \text{Var } B_n^+(un^{2/3}) \rightarrow 8u$ and

$$\lim_{n \rightarrow \infty} B_n^\pm \stackrel{\text{dist.}}{=} B,$$

where B is a standard Brownian Motion. Thus,

$$\limsup_{n \rightarrow \infty} \mathbb{P}(W_{\Delta_n^b}(\delta) > \eta) \leq 2\mathbb{P}(W_B(\delta) > \eta - 3\sqrt{2}\delta r) + R_1(r).$$

If we choose $r = r_\delta := \delta^{-\beta}$, for a fixed $\beta \in (0, 1)$, then

$$\limsup_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}(W_{\Delta_n^b}(\delta) > \eta) \leq 2 \limsup_{\delta \downarrow 0} \mathbb{P}(W_B(\delta) > \eta - 3\sqrt{2}\delta^{1-\beta}) + \limsup_{\delta \downarrow 0} R_1(r_\delta) = 0.$$

We use Lemma 2.3 to control the probability of $E_n(r_\delta)$, while for the other $\limsup_{\delta \downarrow 0}$ we just use that if X is a stochastic process in the space of continuous functions (uniform topology) then

$$\limsup_{\delta \downarrow 0} \mathbb{P}(W_X(\delta) > \eta) = 0.$$

By Theorem 15.5 in [4], this implies tightness of Δ_n^b , and by Prohorov's theorem, this ensure the existence of sub-sequential weak limits.

Given a process X denote

$$X^\epsilon : u \mapsto \epsilon^{1/2} X(\epsilon^{-1}u).$$

By Lemma 2.4, if $|v - u| \leq \delta$ then,

$$B_n^{-,\epsilon}(v) - B_n^{-,\epsilon}(u) - 3\sqrt{2}\epsilon^{1/2}\delta r \leq \Delta_n^{b,\epsilon}(v) - \Delta_n^{b,\epsilon}(u) \leq B_n^{+,\epsilon}(v) - B_n^{+,\epsilon}(u) + 3\sqrt{2}\epsilon^{1/2}\delta r. \quad (2.2)$$

Since $\epsilon^{1/2}B(\epsilon^{-1}x) \stackrel{dist.}{=} B(x)$, if Δ^b is any weak limit of $\{\Delta_n^b : n \geq 1\}$, then

$$\mathbb{P}(W_{\Delta^b, \epsilon}(\delta) > \eta) \leq 2\mathbb{P}(W_B(\delta) > \eta - 3\sqrt{2}\delta\epsilon^{1/2}r) + R_1(r).$$

If we now set $r_\epsilon := \epsilon^{-\beta}$, with $\beta \in (0, 1/2)$, then

$$\limsup_{\epsilon \downarrow 0} \mathbb{P}(W_{\Delta^b, \epsilon}(\delta) > \eta) \leq 2\mathbb{P}(W_B(\delta) > \eta),$$

and hence

$$\limsup_{\delta \downarrow 0} \limsup_{\epsilon \downarrow 0} \mathbb{P}(W_{\Delta^b, \epsilon}(\delta) > \eta) = 0,$$

which implies tightness of $\Delta^{b, \epsilon}$ (in the space of continuous processes). By (2.2),

$$\mathbb{P}\left(\cap_{i=1}^j \left\{\Delta^{b, \epsilon}(u_i) \leq x_i\right\}\right) \leq \mathbb{P}\left(\cap_{i=1}^j \left\{B(u_i) \leq x_i + 3\sqrt{2}\epsilon^{1/2-\beta}\right\}\right) + R_1(r_\epsilon)$$

and

$$\mathbb{P}\left(\cap_{i=1}^j \left\{\Delta^{b, \epsilon}(u_i) \leq x_i\right\}\right) \geq \mathbb{P}\left(\cap_{i=1}^j \left\{B(u_i) \leq x_i - 3\sqrt{2}\epsilon^{1/2-\beta}\right\}\right) + R_1(r_\epsilon),$$

which proves that

$$\mathbb{P}\left(\cap_{i=1}^j \left\{\Delta^b(u_i) \leq x_i\right\}\right) = \mathbb{P}\left(\cap_{i=1}^j \left\{B(u_i) \leq x_i\right\}\right).$$

3 Checking Assumption 1

Proof of Lemma 1.1. Let b be an initial profile and consider the event

$$\left\{Z^b[0]_n \geq u\right\} = \left\{\exists z \in [u, n] : b(z) + L_z[0]_n = L^b[0]_n\right\}.$$

Since

$$L_z[0]_n \leq L^{s_\rho}[0]_n - s_\rho(z) \text{ and } L[0]_n = L_0[0]_n \leq L^b[0]_n$$

(recall that $b(0) = 0$ for the second inequality), we have that

$$\left\{Z^b[0]_n \geq u\right\} \subseteq \left\{\exists z \in [u, n] : s_\rho(z) - b(z) \leq L^{s_\rho}[0]_n - L[0]_n\right\}. \quad (3.1)$$

For the flat profile $b = f \equiv 0$, we can rewrite the event in the right hand side of (3.1) as,

$$a(u) := m(u) + s_\rho(u) \leq L^{s_\rho}[0]_n - L[0]_n,$$

where $m(u)$ is the minimum of $s_\rho(z) - s_\rho(u)$ for $z \geq u$. Hence,

$$\mathbb{P}\left(Z^f[0]_n \geq u\right) \leq \mathbb{P}(a(u) \leq L^{s_\rho}[0]_n - L[0]_n). \quad (3.2)$$

A straightforward computation shows that,

$$\mathbb{E} a(u) = \mathbb{E} m(u) + \frac{2\rho - 1}{\rho(1 - \rho)}u,$$

and (since $m(u)$ and $s_\rho(u)$ are independent)

$$\text{Var } a(u) = \text{Var } m(u) + \left(\frac{1}{(1-\rho)^2} + \frac{1}{\rho^2} \right) u.$$

The random walk $(s(u) - s(u+n), n \geq 0)$ has a negative drift, and its maximum has a well known distribution [?]. This allow us to get that

$$\mathbb{E} m(u) = -\frac{1}{2\rho-1} \frac{1-\rho}{\rho},$$

and

$$\text{Var } m(u) = \frac{2(1-\rho)}{\rho(2\rho-1)^2} - \frac{(1-\rho)^2}{\rho^2(2\rho-1)^2} = \frac{1}{(2\rho-1)^2} \left(2\frac{(1-\rho)}{\rho} - \frac{(1-\rho)^2}{\rho^2} \right).$$

Now, for $c > 0$ (we will set its value later),

$$\begin{aligned} \mathbb{P}(a(u) \leq L^{s_\rho}[0]_n - L[0]_n) &\leq \mathbb{P}\left(L[0]_n - 4n \leq -\frac{r^2 n^{1/3}}{c}\right) \\ &+ \mathbb{P}\left(a(u) \leq L^{s_\rho}[0]_n - 4n + \frac{r^2 n^{1/3}}{c}\right). \end{aligned} \quad (3.3)$$

By Theorem 2.4 in [1], for $\alpha \in (0, 1)$ there exists $c_1 = c_1(\alpha) > 0$ such that for all $n \geq 1$ and $r > 0$,

$$\mathbb{P}\left(L[0]_n - 4n \leq -\frac{r^2 n^{1/3}}{c}\right) \leq \frac{c_1 c^{3\alpha/2}}{(r)^{3\alpha}}.$$

and so, by choosing $\alpha = 3/4$, we get that

$$\limsup_{r \rightarrow \infty} r^{-2} \limsup_{n \rightarrow \infty} \mathbb{P}\left(L[0]_n - 4n \leq -\frac{r^2 n^{1/3}}{c}\right) = 0. \quad (3.4)$$

To estimate the second term in the rhs of (3.3) we do as flows:

$$\begin{aligned} \mathbb{P}\left(a(u) \leq L^{s_\rho}[0]_n - 4n + \frac{r^2 n^{1/3}}{c}\right) &\leq \mathbb{P}\left(L^{s_\rho}[0]_n - 4n \geq \mathbb{E} a(u) - \frac{2r^2 n^{1/3}}{c}\right) \\ &+ \mathbb{P}\left(a(u) \leq \mathbb{E} a(u) - \frac{r^2 n^{1/3}}{c}\right). \end{aligned} \quad (3.5)$$

Write

$$\mathbb{P}\left(L^s[0]_n - 4n \geq \mathbb{E} a(u) - \frac{2r^2 n^{1/3}}{L}\right) = \mathbb{P}\left(L^s[0]_n - \mathbb{E} L^s[0]_n \geq \Lambda - \frac{2r^2 n^{1/3}}{c}\right),$$

where

$$\begin{aligned}
\Lambda &= 4n - \mathbb{E} L^s[0]_n + \mathbb{E} a(u) \\
&= \left(4 - \frac{1}{\rho(1-\rho)}\right)n + \frac{2\rho-1}{\rho(1-\rho)}u + \mathbb{E} m(u) \\
&= \frac{(4\rho(1-\rho)-1)n + (2\rho-1)u}{\rho(1-\rho)} + \mathbb{E} m(u) \\
&\geq \frac{(4\rho(1-\rho)-1)n + (2\rho-1)u}{4} + \mathbb{E} m(u).
\end{aligned}$$

By maximizing the term in the numerator, we get

$$\rho(u, n) = \frac{1}{2} + \frac{u}{4n},$$

and we set $u := rn^{2/3}$. For this choice of $\rho(u, n)$, we have

$$\frac{(4\rho(1-\rho)-1)n + (2\rho-1)u}{4} = \frac{u^2}{16n} = \frac{r^2}{16}n^{1/3}.$$

We still have to control $\mathbb{E} m(u)$:

$$\mathbb{E} m(u) = -\frac{1}{2\rho-1} \frac{1-\rho}{\rho} \geq -\frac{4}{r}n^{1/3}.$$

Thus

$$\Lambda \geq \left(\frac{r^2}{16} - \frac{4}{r}\right)n^{1/3} \geq \frac{r^2}{32}n^{1/3},$$

for $r > 128$. If we choose $c = 128$, then we finally get

$$\begin{aligned}
\mathbb{P}\left(L^s[0]_n - 4n \geq \mathbb{E} a(u) - \frac{2r^2n^{1/3}}{128}\right) &= \mathbb{P}\left(L^s[0]_n - \mathbb{E} L^s[0]_n \geq \Lambda - \frac{r^2n^{1/3}}{64}\right) \\
&\leq \mathbb{P}\left(L^s[0]_n - \mathbb{E} L^s[0]_n \geq \frac{r^2n^{1/3}}{64}\right) \\
&\leq (64)^2 \frac{\text{Var } L^s[0]_n}{r^4n^{2/3}}.
\end{aligned}$$

By Lemma 4.7 in [1],

$$\text{Var } L^{s\rho}[0]_n \leq \frac{\text{Var } L^{s_{1/2}}[0]_n}{4\rho^2} + n \frac{2\rho-1}{\rho^2(1-\rho)^2},$$

together with Theorem 2.1 in [1], this yields to

$$\limsup_{n \rightarrow \infty} \frac{\text{Var } L^{s\rho}[0]_n}{n^{2/3}} \leq \limsup_{n \rightarrow \infty} \frac{\text{Var } L^{s_{1/2}}[0]_n}{n^{2/3}} + 4r \leq c_2 + 4r,$$

for some constant $c_2 > 0$. Therefore,

$$\limsup_{n \rightarrow \infty} \mathbb{P}\left(L^s[0]_n - 4n \geq \mathbb{E} a(u) - \frac{2r^2n^{1/3}}{128}\right) \leq (64)^2 \left(\frac{c_2}{r^4} + \frac{4}{r^3}\right). \quad (3.6)$$

Since for $u = rn^{2/3}$,

$$\lim_{n \rightarrow \infty} \text{Var } X(\rho) = 8 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\text{Var } m(u)}{n^{2/3}} = \frac{1}{16r^2},$$

we have that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \mathbb{P} \left(a(u) \leq \mathbb{E} a(u) - \frac{r^2 n^{1/3}}{128} \right) &\leq \limsup_{n \rightarrow \infty} \frac{(128)^2}{r^4 n^{2/3}} \text{Var } a(u) \\ &= \limsup_{n \rightarrow \infty} \frac{(128)^2}{r^4 n^{2/3}} (\text{Var } m(u) + (\text{Var } X)u) \\ &= \frac{(128)^2}{16r^6} + \frac{8(128)^2}{r^3}. \end{aligned} \quad (3.7)$$

From (3.2), (3.3), (3.4), (3.5), (3.6) and (3.7), we see that

$$\limsup_{r \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P} \left(Z^f[0]_n^+ \geq rn^{2/3} \right) = 0. \quad (3.8)$$

By translation invariance, $Z^f[h]_n \stackrel{\text{dist.}}{=} Z^f[0] + h$, and by symmetry $Z^f[0]_n^- \stackrel{\text{dist.}}{=} Z^f[0]_n^+$, and so (3.8) implies Assumption 1. \square

Proof of Lemma 1.2. The proof of Assumption 1 for mixed profiles is based on the same ideas as in the proof of Lemma 1.1. We just need to adapt a few details in each case.

Wedge to Flat. In that case $Z^{\text{wf}}[k]_n \geq 0$. By (3.2),

$$\mathbb{P} \left(Z^{\text{wf}}[\pm n^{2/3}]_n \geq u \right) \leq \mathbb{P} \left(a(u) \leq L^{s_p}[\pm n^{2/3}]_n - L[\pm n^{2/3}]_n \right),$$

with $a(u)$ as define in the proof of Lemma 1.1. We have that

$$\limsup_{r \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P} \left(L[\pm n^{2/3}]_n - 4n \leq -\frac{r^2 n^{1/3}}{128} \right) = 0. \quad (3.9)$$

There are two ways to justify (3.9). One is by proving (3.6) for the probability above with the help of the method developed in [1]. We do believe this is possible, but we will not do it here. Another possibility is to directly use (marginal) convergence to Airy_2 process (minus a quadratic drift) and the tail behavior of the limiting distribution (Tracy-Widom).

Now, for the same choice of $\rho(u, n)$ as in Lemma 1.1, this leave us with (recall the proof of (3.5),

(3.6) and (3.7))

$$\begin{aligned}
\mathbb{P} \left(a(u) \leq L^{s_\rho}[\pm n^{2/3}]_n - 4n + \frac{r^2 n^{1/3}}{128} \right) &\leq \mathbb{P} \left(L^{s_\rho}[\pm n^{2/3}]_n - \mathbb{E} L^{s_\rho}[0]_n \geq \frac{r^2}{64} n^{1/3} \right) \\
&+ \mathbb{P} \left(a(u) \leq \mathbb{E} a(u) - \frac{r^2 n^{1/3}}{128} \right) \\
&= \mathbb{P} \left(L^{s_\rho}[0]_n - \mathbb{E} L^{s_\rho}[0]_n + s_\rho(\pm n^{2/3}) \geq \frac{r^2}{64} n^{1/3} \right) \\
&+ \mathbb{P} \left(a(u) \leq \mathbb{E} a(u) - \frac{r^2 n^{1/3}}{128} \right) \\
&= \mathbb{P} \left(L^{s_\rho}[0]_n - \mathbb{E} L^{s_\rho}[0]_n \geq \frac{r^2}{128} n^{1/3} \right) \\
&+ \mathbb{P} \left(s_\rho(\pm n^{2/3}) \geq \frac{r^2}{128} n^{1/3} \right) \\
&+ \mathbb{P} \left(a(u) \leq \mathbb{E} a(u) - \frac{r^2 n^{1/3}}{128} \right),
\end{aligned}$$

where we use that $L^s[\pm n^{2/3}]_n \stackrel{dist.}{=} L^s[0]_n + s_\rho(\pm n^{2/3})$. The only term that did not appear in the proof of Lemma 1.1 is

$$\mathbb{P} \left(s_\rho(\pm n^{2/3}) \geq \frac{r^2}{128} n^{1/3} \right) = \mathbb{P} \left(s_\rho(\pm n^{2/3}) - \mathbb{E} s_\rho(\pm n^{2/3}) \geq \frac{r^2}{128} n^{1/3} - \mathbb{E} s_\rho(\pm n^{2/3}) \right).$$

But since s_ρ is a random walk with drift

$$\mathbb{E} s_\rho(\pm n^{2/3}) = \frac{2\rho - 1}{\rho(1 - \rho)} n^{2/3} \sim r n^{1/3},$$

it is not hard to see that

$$\limsup_{r \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P} \left(s_\rho(\pm n^{2/3}) \geq \frac{r^2}{128} n^{1/3} \right) = 0.$$

Wedge to Stationary. Again we have $Z^{\text{ws}}[k]_n \geq 0$ and, for $\rho > 1/2$,

$$\mathbb{P} \left(Z^{\text{ws}}[\pm n^{2/3}]_n \geq u \right) \leq \mathbb{P} \left(a(u) \leq L^{s_\rho}[\pm n^{2/3}]_n - L[\pm n^{2/3}]_n \right),$$

but now $a(u)$ has a different form:

$$a(u) := m(u) + t_\rho(u),$$

where

$$t_\rho(u) := s_\rho(u) - s_{1/2}(u)$$

and $m(u)$ is the minimum of the random walk

$$(s_\rho(u+n) - s_{1/2}(u+n)) - t_\rho(u) \text{ for } n \geq 0. \quad (3.10)$$

Following [1], we couple

$$s_\rho(k) = \sum_{i=1}^k \bar{\zeta}_i \text{ with } s_{1/2}(k) = \sum_{i=1}^k \zeta_i,$$

by taking

$$\zeta_i := \text{Exp}_{1,i}(1/2) - \text{Exp}_{2,i}(1/2) \text{ and } \bar{\zeta}_i := \frac{1}{2(1-\rho)} \text{Exp}_{1,i}(1/2) - \frac{1}{2\rho} \text{Exp}_{2,i}(1/2).$$

The random walk (3.10) will have increments

$$\frac{2\rho-1}{2(1-\rho)} \text{Exp}_{1,i}(1/2) + \frac{2\rho-1}{2\rho} \text{Exp}_{2,i}(1/2), \text{ for } i \geq 1,$$

and thus $m(u) = 0$. Therefore, we are left with

$$\mathbb{P}\left(Z^{\text{ws}}[\pm n^{2/3}]_n \geq u\right) \leq \mathbb{P}(t_\rho(u) \leq L^{s_\rho}[\pm n^{2/3}]_n - L[\pm n^{2/3}]_n).$$

We can apply (3.9) again and get an inequality similar to (3.3). We note that $t_\rho(u)$ has the same mean as $s_\rho(u)$, so similar estimates can be done to get the analog of (3.6). Since, $2\rho-1 = u(4n)^{-1}$ and $u = rn^{2/3}$,

$$\text{Var } t_\rho(u) \leq c_3(2\rho-1)^2 u \sim \frac{u^3}{n^2} \sim r^3,$$

we get the analog of (3.7) but with $\limsup_n (\cdot) = 0$.

Flat to Stationary. We can estimate $Z^{\text{fs}}[k]_n^-$ by doing the same reasoning applied to $Z^{\text{wf}}[k]_n$. For $k < 0$, the stationary measure has increment $\text{Exp}_1(\rho) - \text{Exp}_2(1-\rho)$, and we will have to chose $\rho(u, n) = 1 - u(4n)^{-1}$ instead. For $Z^{\text{fs}}[k]_n^+$ we can proceed exactly as we did to analyse $Z^{\text{ws}}[k]_n$. \square

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